# Calculus 1 in a nutshell

## Formulas & Summary

Pi Pinnacle Tutors Ltd.

# Log and Ln rules

$$ln(x) = log_e(x)$$

$$log_a(b) = \frac{log_{c(b)}}{log_{c(a)}}$$

$$log_a(a^x) = x$$

$$ln(e^x) = x$$

$$e^{ln(x)} = x$$

$$a^{log_a(x)} = x$$

$$log_a(x^n) = n\, log_a(x)$$

$$ln(x^n) = n\, ln(x)$$

$$log_a(xy) = log_a(x) + log_a(y)$$

$$log_a\left(\frac{x}{y}\right) = log_a(x) - log_a(y)$$

$$ln(xy) = ln(x) + ln(y)$$

$$ln\left(\frac{x}{y}\right) = ln(x) - ln(y)$$

## Trigonometry

$$sin(x) = \frac{1}{csc(x)}$$

$$cos(x) = \frac{1}{sec(x)}$$

$$tan(x) = \frac{sin\ (x)}{cos(x)}$$

$$csc(x) = \frac{1}{sin\ (x)}$$

$$sec(x) = \frac{1}{cos(x)}$$

$$cot(x) = \frac{cos\ (x)}{sin(x)}$$

$$sin^2(x) + cos^2(x) = 1$$

$1 + cot^2(x) = csc^2(x)$, (divide each term of formula above by $sin^2(x)$)

$1 + tan^2(x) = sec^2(x)$, (divide each term of formula above by $cos^2(x)$)

## Algebra

Domain: All possible x-values the function can take that produces a valid y-value.

Range: All possible y-values the function can take that produces valid x-values.

$a^2 - b^2 = (a+b)(a-b)$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a+b)^2 = a^2 + 2ab + b^2$

$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$\sqrt{xy} = \sqrt{x}\sqrt{y}, x \geq 0 \; and \; y \geq 0$

$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}, x \geq 0 \; and \; y > 0$

$\sqrt{x^2} \neq x, \;\; \sqrt{x^2} = |x|$ (important)

$(a^n)^m = a^{m \cdot n}$

$a^n \cdot b^n = (ab)^n$

$a^n \cdot a^m = a^{m+n}$

$\frac{a^n}{a^m} = a^n \cdot a^{-m}$

# Limits & Continuity

## What is a limit, really?

Let us start by explaining a limit in layman's terms. The limit of a function is the value of the **y-axis** as a function approaches and x-value. It does not matter if the function reaches that limit or not.

## One sided and two-sided limits

One-sided limits are the limits from either side. The left-hand limit is the value the function approaches as we come from the left (negative side). The right-hand limit is the value the function approaches as we come from the right (positive side). Think about the number line. From left to right, it starts with $-\infty$ and ends with $+\infty$.

## Limit mathematically

The limit at a point x=a, denoted as $\lim_{x \to a} f(x)$ exists if and only if the three conditions below are true:

1. The left-hand limit exists at x=a, that is $\lim_{x \to a-} f(x)$ exists.
2. The right-hand limit exists at x=a, that is $\lim_{x \to a+} f(x)$ exists.
3. Condition 1 and 2 have the same value, meaning that the right and left sided limits are equal.

   The limit does not exist if **one or more** of the conditions above is violated.

## Continuity

The easiest way to define continuity is, if you can draw a graph of a function without ever lifting your pencil off the sheet from left to right, then the function is continuous. If we ever have to lift our pencil to keep graphing the function at a certain point/points, there must be a discontinuity and that point(s).

## Types of discontinuities

1. **Jump discontinuity**

A jump discontinuity happens when there's a big gap in the graph, not caused by an asymptote. This often occurs in piecewise functions. The overall limit does not exist at a jump discontinuity.

2. **Infinite discontinuity**

An infinite discontinuity happens where there's an asymptote. It occurs in rational functions when the denominator becomes zero due to factors that can't be cancelled out.

3. **Endpoint discontinuity**

Endpoint discontinuities occur at the start (a) and end (b) of a function defined over an interval [a, b]. The overall limit does not exist at these endpoints.

4. **Hole discontinuity**

This type of discontinuity is the most uncommon one, A hole discontinuity happens when there's a missing point in the graph, often because a factor in the numerator and denominator cancels out. For example, in the function $f(x) = \frac{(x-1)(x+1)}{x-1}$, there's a hole at x=1. Be careful, the functions $f(x) = \frac{(x-1)(x+1)}{x-1}$ $g_{(x)} = x + 1$ are not equal! They are equal everywhere, except at x=1.

## Solving Limits

These are the five main ways to solve limits:

1. Direct substitution - To find the limit of a function as xxx approaches a certain value, you can often just plug that value into the function. If the function gives you a real number, that is the limit.
2. Factoring - If direct substitution gives an indeterminate form like $\frac{0}{0}$. $\frac{\infty}{\infty}$, $1^{\infty}$, $0 \cdot \infty$, etc., try to factor the numerator and the denominator to cancel out common factors, then substitute again.
3. Conjugate- For limits involving square roots, multiply by the conjugate (change the sign between terms) to simplify the expression and remove the square root.
4. Applying ln() to the limit- For limits involving exponential expressions (neither of the expressions is a constant), apply the natural logarithm (ln) to simplify. After finding the limit of the ln, exponentiate the result to get the final limit.
5. L'Hôpital's rule - f direct substitution gives an indeterminate form like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ only, use L'Hôpital's Rule: take the derivatives of the numerator and the denominator, then substitute again.

## Special limits-Absolute value

1. Identify the points where the function inside the absolute value is zero
2. Compute two limits, one with $x \to a^-$ and $x \to a^+$
3. Solve and compare the limits, if they are equal, the limit exists. It does not otherwise

Example:

$$\lim_{x \to 1} \frac{|x-1|}{x^2-1}$$

1. $|x-1|$ is 0 when x=1.
2. If $x \to 1^-$: $\lim_{x \to 1^-} \frac{|x-1|}{x^2-1} = \lim_{x \to 1^-} \frac{-(x-1)}{x^2-1} = \lim_{x \to 1^-} \frac{-(x-1)}{(x-1)(x+1)} = \lim_{x \to 1^-} - \frac{1}{x+1} = -\frac{1}{2}$

   If $x \to 1^+$: $\lim_{x \to 1^+} \frac{|x-1|}{x^2-1} = \lim_{x \to 1^+} \frac{(x-1)}{x^2-1} = \lim_{x \to 1^+} \frac{(x-1)}{(x-1)(x+1)} = \lim_{x \to 1^+} \frac{1}{x+1} = \frac{1}{2}$
3. Since $\frac{1}{2} \neq -\frac{1}{2}$, the limit does not exist.

## Limits at infinity

A limit is infinite or also called divergent when the value of the limit is ∞ or -∞ as x approaches a particular point.

## Asymptotes

Asymptotes are lines that a graph approaches but never actually touches (always true for vertical asymptotes). There are two main types of asymptotes: vertical and horizontal. Understanding these helps in analyzing the behavior of functions and helps us to sketch them.

Vertical asymptotes - A vertical asymptote is a vertical line x=a where the function grows without bound as x approaches a. This often happens when the denominator of a rational function is zero at x=a and can't be canceled out. However, vertical asymptotes can also occur in other functions, not just rational ones. Some examples include, but are not limited to functions containing ln() or log(), trigonometric functions, piecewise functions.

Horizontal asymptotes - A horizontal asymptote is a horizontal line y=b where the function approaches b as x goes to ∞ or -∞. Horizontal asymptotes describe the long-term behavior of a function. Note that in some rare instances, a function can cross a horizontal asymptote but eventually cross it back and never cross it again.

Special case for rational functions involving polynomials: If we have a function of the type: $f(x) = \frac{a_n x^n + a_{n-1} x_+^{n-1} \ldots a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots b_1 x + b_0}$, where all the $\{a_n, \ldots, a_0\}$ and $\{b_m, \ldots, b_0\}$ are real numbers and $m$ and $n$ are integers, then we have the quotient of two polynomials. If this is the case,

$\lim\limits_{x \to \infty} \frac{a_n x^n + a_{n-1} x_+^{n-1} \ldots a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots b_1 x + b_0}$ and $\lim\limits_{x \to -\infty} \frac{a_n x^n + a_{n-1} x_+^{n-1} \ldots a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots b_1 x + b_0}$ are equal to:

- $\frac{a}{b}$ if $m = n$, horizontal asymptote $y = \frac{a}{b}$
- $0$ if $m > n$, horizontal asymptote $y = 0$
- ∞ or -∞ if $m < n$, horizontal asymptote does not exist.

# Derivatives

A derivative represents the rate at which a function is changing at any given point. In simple terms, it's like finding the slope of the function at a specific point. It tells us how fast the y-value of a function is changing as the x-value changes.

To fully grasp derivatives, we must first understand limits. The derivative at a point is defined as the limit of the average rate of change (slope of the secant line) of the function as the interval approaches zero. This means we look at how the function behaves very close to that point.

## Derivatives mathematically

Two usual mathematical definitions are used to describe a derivative: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ or $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$. Notice that in both cases, we are calculating the slope at a single point (two points that are infinitely close), also called the instantaneous rate of change.

### The chain rule:

This is perhaps the most important aspect of derivatives. At the most elementary levels of derivatives, we are performing a chain rule. The chain rule is used for calculating derivatives of nested functions, one function is the outside function, and one is the inside function.

If $y = f\,(g(x))$, $y' = f'\big(g(x)\big)\cdot g'(x)$. Basically, first identify $f(x)$ and $g(x)$. Then take the derivative of the outside function, leaving the inside untouched, then multiply that result by the derivative of the inside function.

We will see the derivatives formula sheet soon, but if we take the power rule for example to better understand that we are always doing a chain rule, let us take the example of

$y = (x^2+1)^3$. In this case, our outside function is $x^3$ and our inside function is $(x^2+1)$. Therefore, $f'\big(g(x)\big) = 3(x^2+1)^2$ and $g'(x) = (2x)$. Combining them, $y' = 6x(x^2+1)^2$. In the case of just having $y = x^3$, the inside function is only $x$. So the derivative becomes: $f'\big(g(x)\big) = 3x^2$ and $g'(x) = (x)' = 1$.

## Derivatives rules

| Name | Derivative | Remarks |
|---|---|---|
| Power rule<br>$ax^n$ | $nax^{n-1}$ | If we had $a[f(x)]^n$,<br>Derivative<br>= $na[f(x)]^{n-1} \cdot f'(x)$<br><br>Square roots and other roots are the power rule,<br><br>$\sqrt[b]{f^a(x)} = [f(x)]^{\frac{a}{b}}$<br>If $n < 0$, use the law of exponents, $\frac{a}{x^n} = ax^{-n}$ and use the power rule |
| Constant<br>$a$ | 0 | / |
| Product rule<br>$f(x)g(x)$ | $f'(x)g(x) + f(x)g'(x)$ | / |
| Quotient rule<br>$\frac{f(x)}{g_{(x)}}$ | $\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ | We can also use a product rule,<br>$\frac{f(x)}{g(x)} = f(x) \cdot [g(x)]^{-1}$<br><br>If $f(x) = a$ (constant function),<br>Derivative= $\frac{-ag'(x)}{[g(x)]^2}$ |
| Exponential rule<br><br>$a^{f(x)}$ | $a^{f(x)} \cdot f'(x) \cdot ln(a)$ | $a \neq 1$<br>$a > 0$ |
| Power of two functions<br>$f(x)^{g(x)}$ | $f(x)^{g(x)}\left(g'(x)\,ln\big(f(x)\big) + \ g(x)\frac{f'(x)}{f(x)}\right)$ | Also called logarithmic differentiation |
| Ln rule<br>$ln(x)$ | $\frac{1}{x}$ | If we had $ln\big(f(x)\big)$,<br>Derivative<br>=$\frac{f'(x)}{f(x)}$ |
| Log rule<br>$log_a(x)$ | $\frac{1}{x\,ln(a)}$ | If we had<br>$log_a(f(x))$,<br>Derivative = $\frac{f'(x)}{f(x) \cdot ln(a)}$<br>$a > 0$<br>$a \neq 1$ |
| Sum and Difference rules<br>$f(x) \pm g(x)$ | $f'(x) \pm g'(x)$ | / |

## Other derivatives

| Function | Derivative |
| --- | --- |
| $sin(f(x))$ | $cos(f(x)) \cdot f'(x)$ |
| $cos(f(x))$ | $-sin(f(x)) \cdot f'(x)$ |
| $tan(f(x))$ | $sec^2(f(x)) \cdot f'(x)$ |
| $cot(f(x))$ | $-csc^2(f(x)) \cdot f'(x)$ |
| $sec(f(x))$ | $tan(f(x)) \cdot sec(f(x)) \cdot f'(x)$ |
| $csc(f(x))$ | $-cot(f(x)) \cdot csc(f(x)) \cdot f'(x)$ |
| $arcsin(f(x))$ | $\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$ |
| $arccos(f(x))$ | $-\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$ |
| $arctan(f(x))$ | $\frac{f'(x)}{1+[f(x)]^2}$ |
| $arccot(f(x))$ | $-\frac{f'(x)}{1+[f(x)]^2}$ |
| $arcsec(f(x))$ | $\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$ |
| $arccsc(f(x))$ | $-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{[f(x)]^2-1}}$ |

## Tangent and normal lines

A tangent line is a line that touches the graph of a function at exactly one point, the point x = a.

$$y = f'(a)(x - a) + f(a)$$

The normal line to a function at a specific point is perpendicular to the tangent line at that point. If the slope of the tangent line is mmm, then the slope of the normal line is the negative reciprocal of $m$, which is $-\frac{1}{m}$.

## Finding the equation of the normal line

1. Take the derivative of the original function and evaluate it at the given point. This gives you the slope of the tangent line, which we'll call $m$.
2. Find the negative reciprocal of $m$, which is $-\frac{1}{m}$. This is the slope of the normal line, which to avoid confusion we will call $n$; $n = -\frac{1}{m}$.
3. Plug $n$ and the given point into the point-slope formula for the equation of the line, $(y - y_1) = n(x - x_1)$.
4. Solve for $y$.

## Average rate of change (slope formula)

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

## Differentiability and Continuity

A function that is differentiable at a point or an interval, is continuous at that point or interval. The opposite is not necessarily true! A function can be continuous everywhere but still not differentiable at some points or intervals.

1. A piecewise function will be continuous at its break point if the left-hand limit and the right-hand limit at that point are equal.

2. A piecewise function will be smooth at its break point if the left-hand limit and the right-hand limit of the slopes (the value of the derivative) at that point are equal.

## Implicit differentiation

Implicit differentiation allows us to find the derivative of a function where both $x$ and $y$ are on the same side of the equation, meaning $y$ cannot be isolated **totally**.

To use implicit differentiation:

1. Differentiate both sides with respect to $x$.
2. When we see $y$, we differentiate it like we would $x$, but we multiply that term by the derivative of $y$, which we express as $y'$ or $\frac{dy}{dx}$.
3. Move all terms that contain $\frac{dy}{dx}$ to left side and everything else to the right.
4. Factor $\frac{dy}{dx}$ on the left side and divide both sides by the other left side factor so that $\frac{dy}{dx}$ is isolated on the left side.

To find the tangent line to an implicit function, follow the same steps as the previous page once that $\frac{dy}{dx}$ is isolated on the left side.

## Higher-order derivatives

The first derivative is the value we find when we initially take the derivative of a function. The second derivative is the derivative of the first derivative; it's what we get when we take the derivative of the derivative.

| **First Derivative** | **Second derivative** | **n^th derivative** |
|---|---|---|
| $y'$ | $y''$ | / |
| $f'(x)$ | $f''(x)$ | $f^n(x)$ |
| $\frac{dy}{dx}$ | $\frac{d^2y}{dx^2}$ | $\frac{d^ny}{dx^n}$ |

# Theoretical and real-world applications of derivatives

## Optimization and graphs sketching

### Extrema

Extrema refer to the **$y$-values** “Extreme points”. This can include:

- Global (absolute) maximum: The highest value that the function reaches. There can only be one global maximum, but the function can achieve this maximum at multiple points.

- Global (absolute) maximum: The lowest value that the function reaches. There can only be one global minimum, but the function can achieve this minimum at multiple points.

- Local (relative) maximum: The highest value that the function reaches in a specific region of the graph(locally). There can be an infinite number of local maxima, each with different values and occurring at different points.

- Local (relative) minimum: The lowest value that the function reaches in a specific region of the graph(locally). There can be an infinite number of local minima, each with different values and occurring at different points.

A local or relative maximum occurs when a function changes from increasing to decreasing. If this local maximum is also the highest point in the entire domain of the function, it is called the global or absolute maximum. While a function can have an infinite number of local or relative maxima, it will only have one or no global or absolute maximum.

Similarly, a local or relative minimum happens when a function changes from decreasing to increasing. If this local minimum is also the lowest point in the entire domain of the function, it is known as the global or absolute minimum. A function can have an infinite number of local or relative minima, but it will have only one or no global or absolute minimum.

## Critical points

Critical points occur where the derivative is equal to zero, or where the derivative is undefined. These points indicate where the graph of the function changes direction, either from decreasing to increasing or vice versa. Since the function changes direction at critical points, there will always be at least a local maximum or minimum at these points, and sometimes a global maximum or minimum as well.

## Increasing and decreasing

When the derivative is positive, the function is increasing, meaning it moves upward as we go from left to right. Conversely, when the derivative is negative, the function is decreasing, which means it moves downward as we go from left to right.

## Using the first derivative

- If the derivative is negative to the left of a critical point and positive to the right of it, the graph has a local minimum at that point. It's also possible that this local minimum could be a global minimum.

- If the derivative is positive to the left of a critical point and negative to the right of it, the graph has a local maximum at that point. It's also possible that this local maximum could be a global maximum.

## Inflection points

A point where the function changes from concave up to concave down, or from concave down to concave up, is known as an inflection point. The second derivative is zero at those points.

## Concavity

- The function is concave up when the second derivative is larger than zero, it is said that the function is concave up when it is “smiling”, like a bowl

- The function is concave down when the second derivative is smaller than zero, it is said that the function is concave down when the function is “sad”, like a dome.

## Using the second derivative

- If $f''(x) > 0$ at a **critical point,** a local **minimum** exists there.
- If $f''(x) < 0$ at a **critical point,** a local **maximum** exists there.

## Intercepts

$x - intercept$ -> the $y$ value is 0

$y - intercept$ -> the $x$ value is 0

## Interaction between $f(x)$, $f'(x)$ and $f''(x)$ when **sketching graphs**

| $f(x)$ | $f'(x)$ | $f''(x)$ |
|---|---|---|
| Critical point | $0(x - intercept)$ | / |
| Increasing | Positive (above $x - axis$) | / |
| Decreasing | Negative (below $x - axis$) | / |
| Inflection point | Critical point | $0(x - intercept)$ |
| Concave up | Increasing | Positive (above $x - axis$) |
| Concave down | Decreasing | Negative (below $x - axis$) |
| / | Inflection point | Critical point |
| / | Concave up | Increasing |
| / | Concave down | Decreasing |

## Linear approximation

When we use the tangent line equation as an approximation tool, we refer to it as the linear approximation (or linearization) equation, rather than the tangent line equation.

$$L(x) = f'(a)(x - a) + f(a)$$

## Taylor series

Taylor series are used to approximate complex functions with polynomials around a specific value $a$. By expressing a function as an infinite sum of its derivatives at $a$, Taylor series provide a powerful method for approximation and analysis. A special case of the Taylor series, called the Maclaurin series, occurs when $a$=0.

The Taylor series around the value $x = a$ up until the n$^{th}$ derivative is:

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^k(a)}{k!}(x-a)^k$$

Where $f^k(a)$ is the k$^{th}$ derivative of $f(x)$ evaluated at $x = a$. Notice the linear approximation described above is the Taylor series when $n = 1$.

## Absolute error

The absolute error for an estimation at a particular point $a$ is the absolute value of the difference between the function's actual value and the linear approximation at that point.

Absolute error = $|f(x) - L(a)|$

## Relative error

Relative error is the amount of error in the approximation compared to the actual value of the function.

Relative error = $\frac{\text{Absolute error}}{f(a)} = \frac{|f(x)-L(a)|}{f(a)}$

## Related Rates

In general, to solve a related rates problem we follow the following framework:

1. Build an equation containing all the relevant variables, solving for some of them using other information if necessary.
2. Implicitly differentiate the equation with respect to time $t$ before plugging in any of the known values.
3. Plug in all the known values, leaving only the variable we are trying to solve for.
4. Solve for the unknown variable.

Common formulas we need for related rates problems:

Cube: $V = s^3$

Pyramid: $V = \frac{1}{3}s^2h$

Cone: $V = \frac{1}{3}\pi r^2 h$

Triangular prism: $V = \frac{1}{2}wlh$

Rectangular prism: $V = wlh$

Sphere: $V = \frac{4}{3}\pi r^3$

Cylinder: $V = \pi r^2 h$

Pythagorean theorem: $a^2 + b^2 = c^2$

## Optimization

In general, to solve an optimization problem we follow the following framework:

1. Formulate an equation in a single variable that represents the value we aim to maximize or minimize.
2. Take the derivative and set it equal to zero to find the critical points. Then, use the first derivative test to determine where the function is increasing and decreasing.
3. Based on the increasing and decreasing behavior of the function, identify its maxima and minima.
4. Use the extrema to answer the question at hand.

## Theorems to know

**Mean Value Theorem:** The Mean Value Theorem states that if a function is continuous (unbroken) and differentiable (smooth) throughout the chosen interval, then there must be a point within the interval where the tangent to the curve is parallel to the line connecting the endpoints.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The consequence of the Mean Value Theorem is that the instantaneous rate of change at $x = c$ will be equal to the average rate of change over the interval.

**Rolle's Theorem:** Rolle's Theorem states that if a function is continuous (unbroken) and differentiable (smooth) within an interval, there must be a tangent line parallel to the horizontal line connecting the endpoints.

Rolle's Theorem can demonstrate the following:

- The presence of a horizontal tangent line within the interval.
- A point where the derivative is zero in the interval.
- The existence of a critical point in the given interval.
- A point within the interval where the function changes direction, either from increasing to decreasing or from decreasing to increasing.

**Intermediate value theorem:** If $f(x)$ is continuous on the interval $[a, b]$ and $y_0$ is a number between $f(a)$ and $f(b)$, then there is at least one point c in $(a, b)$ where $f(c)$= $y_0$. This means the function reaches every value between $f(a)$ and $f(b)$.

### L'Hôpital's rule

If direct substitution into the function gives the following indeterminate forms: $\frac{\pm\infty}{\pm\infty}, \frac{0}{0}$ or $\pm\infty$ $(requires\ manipulation\ first)$, we apply L'Hôpital's rule by replacing both the numerator and the denominator of the fraction with their respective derivatives.

## Physics

| | |
|---|---|
| Position | $x(t)$ |
| Velocity | $v(t) = x'(t)$(can be positive or negative) |
| Acceleration | $a(t) = v'(t) = x''(t)$(can be positive or negative) |

### Speed

Speed is always positive; it has no direction.

$$s(t) = |v(t)|$$

Speed **increases** if and only if the velocity and acceleration have the same sign,

$$v(t) \cdot a(t) > 0$$

Speed **decreases** if and only if the velocity and acceleration have the opposite sign,

$$v(t) \cdot a(t) < 0$$

### Velocity

Velocity can be positive or negative, depending on the direction of the object.

- Object moving to the right (forward), $v(t) > 0$
- Object moving to the left (backward), $v(t) < 0$
- Object is not moving (at rest), $v(t) = 0$
- Velocity is increasing when $a(t) > 0$
- Velocity is decreasing when $a(t) < 0$

### Vertical Motion

If units are expressed in feet, use $g = 32.15\frac{ft}{s^2}$ for the gravitational constant, otherwise use $g = 9.8\frac{m}{s^2}$ for when the units are in meters.

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$

### Average and instantaneous velocity

To find instantaneous velocity, we simply evaluate the velocity function $v(t)$ at $t = a$. To find the average velocity:

$$\Delta v(a,b) = \frac{x(b) - x(a)}{b - a}$$

## Finance

If $p$ is the demand function for a product, $F$ is fixed cost and $V(x)$ is the variable cost:

| | |
|---|---|
| Revenue | $R(x) = xp$ |
| Cost | $F + V(x)$ |
| Profit | $R(x) - C(x)$ |
| Marginal Revenue | $R'(x)$ |
| Marginal Cost | $C'(x)$ |
| Marginal Profit | $R'(x) - C'(x)$ |

Maximum profit occurs when the marginal revenue and marginal cost are equal: $R'(x) = C'(x)$.

Production level that minimizes the average cost: $C'(x) = \frac{C(x)}{x}$.

## Exponential growth and decay

The basic equation for exponential decay is $y = Ce^{kt}$,

Where $C$, represents the initial amount of a substance, $k$ is the decay constant and $y$ is the amount of substance we have remaining after time $t$.

### Half life

The half-life is the amount of time required for exactly half of the original substance to decay, leaving exactly half of what we started with.

### Newton's Law of Cooling

Newton's Law of Cooling models how a warm object in a cooler environment cools down until it reaches the temperature of its surroundings. The formula is: $\frac{dT}{dt} = -k(T - T_a)$, with $T(0) = T_0$ and $T_a$ being the ambient temperature.