Calculus 2 in a nutshell

Formulas & Summary

Pi PinnacleTutors Ltd.

Techniques of Integration

U-Substitution

This is used to undo the chain rule for differentiation. Here are the steps:

- Identify the inner function, a function whose derivative also appears in the integrand and set it equal to u.
- Compute $du = g'(x)\,dx$.
- Rewrite the integral in terms of u, there should not be any x. Sometimes, when performing the initial u-substitution, we might need to express x's in terms of u and vice-versa. For example, if $u = ln(x) \longrightarrow e^u = x$, if $u = x - 2 \longrightarrow\ u + 3 = x + 1$.
- Integrate with respect to u and substitute all the x's back.

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

Always select u in this order (LIATE): Logarithmic, Inverse Trigonometric, Algebraic expression, Trigonometric and Exponential, then set dv with to remaining part. Best used when we have a polynomial that multiplies a trigonometric function, exponential or logarithmic.

A special case – boomerang

The boomerang case in integration by parts happens when applying the integration by parts formula to an integral eventually brings you back to the original integral. This apparent loop can be used to solve the integral through algebraic manipulation. This case is common with trigonometric functions and exponential functions together.

Trigonometric integrals

Use trigonometric identities/formulas like the ones below to simplify the integral as much as possible. If we have a product of sines and cosines:

$$\int \sin^n(x)\cos^m(x)dx$$

If the exponent on the sine functions (n) is odd, we can extract one sine term and convert the remaining terms to cosines using $\sin^2(x) + \cos^2(x) = 1$, and then use $u = \cos(x)$. Similarly, if m is odd, take out one cosine and convert the rest of the sines with $u = \sin(x)$. If n and m are odd, convert the term with the smallest exponent.

if n and m are even, most of the time those types of integrals can be solved using the 6th, 7th or 8th formula below. By reducing these integrals, sometimes we will run across products of sine and cosine in which the arguments are different. In that case, we should use the formulas 12th, 13th or 14th below.

What if we have a product of tangent and secants?

$$\int \sec^n(x)\tan^m(x)dx$$

If we use the substitution $u = \tan(x)$, we need two secants remaining for the substitution to work. Therefore, if the exponent on the secant (n) is even, we can factor out two secants and then convert the remaining secants to tangents using equation 11 below. Alternatively, if we use the substitution $u = \sec(x)$, we need one secant and one tangent left for the substitution. Thus, if the exponent on the tangent (m) is odd and there is at least one secant in the integrand, we can extract one tangent and one secant. This will make the exponent on the tangent even, allowing us to use equation 11 below to convert the remaining tangents to secants. Note that this method requires at least one secant in the integral. If there are no secants, a different approach is needed.

When the exponent on the secant is even and the exponent on the tangent is odd, either method can be used. However, it is generally easier to convert the term with the smallest exponent.

1. $\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx,\ n \neq 1$
2. $\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx,\ n \neq 1$
3. $\int \cot^n(x)\,dx = -\frac{\cot^{n-1}(x)}{n-1} - \int \cot^{n-2}(x)\,dx,\ n \neq 1$
4. $\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$
5. $\int \cos^n(x)\,dx = \frac{1}{n}\sin(x)\cos^{n-1}(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$
6. $\cos^2(x) = \frac{1+\cos(2x)}{2}$
7. $\sin^2(x) = \frac{1-\cos(2x)}{2}$
8. $\sin(2x) = 2\sin(x)\cos(x)$
9. $\sin^2(x) + \cos^2(x) = 1$
10. $1 + \cot^2(x) = \csc^2(x)$
11. $1 + \tan^2(x) = \sec^2(x)$
12. $\sin(a)\cos(b) = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$
13. $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$
14. $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

Trigonometric substitution

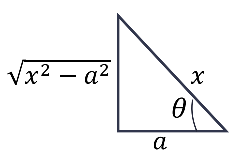
If we have $\sqrt{kx^2 \pm a^2}$ or $\sqrt{a^2 - kx^2}$ where $k > 0$ is a constant. Then:

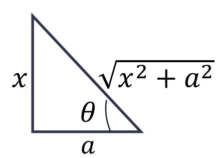
$$\sqrt{kx^2 \pm a^2} = \sqrt{k}\sqrt{x^2 \pm \frac{a^2}{k}} \text{ and } \sqrt{a^2 - kx^2} = \sqrt{k}\sqrt{\frac{a^2}{k} - x^2}$$

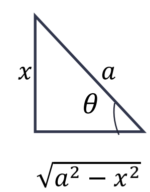
In some rare cases, we might have $\sqrt{ax^2 + bx + c}$ on the denominator, in that case, we must complete the square; $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$

	sec	tan	sin
Integral contains	$\sqrt{x^2 - a^2}$	$\sqrt{x^2 + a^2}$	$\sqrt{a^2 - x^2}$
Substitute	$x = a\sec(\theta)$	$x = a\tan(\theta)$	$x = a\sin(\theta)$
Use identity	$sec^2(\theta) - 1 = tan^2(\theta)$	$sec^2(\theta) = tan^2(\theta) + 1$	$1 - \sin^2(\theta) = \cos^2(\theta)$
Express trigonometric with x	$\sec(\theta) = \frac{x}{a}$	$\tan(\theta) = \frac{x}{a}$	$\sin(\theta) = \frac{x}{a}$
$d\theta$	$dx = a\sec(\theta)\tan(\theta)\,d\theta$	$dx = a\sec^2(\theta)$	$dx = a\cos(\theta)$
θ	$\theta = \text{arcsec}\left(\frac{x}{a}\right)$	$\theta = \arctan\left(\frac{x}{a}\right)$	$\theta = \arcsin\left(\frac{x}{a}\right)$

Triangle







Partial Fractions

When an integral is a fraction of the form: $\frac{a_n x^n + a_{n-1} x^{n-1}_{+} \ldots a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots b_1 x + b_0}$ with $n < m$ (if $m = n$, perform long division) and $b_m x^m + b_{m-1} x^{m-1} + \cdots b_1 x + b_0$ (the denominator) can be factored into linear and/or **irreducible** quadratic factors.

First, completely factor the denominator and:

- For each distinct linear factor $(ax + b)$, set the fraction above equal to $\frac{A}{a_1 x + b_1} + \frac{B}{a_2 x + b_2} + \cdots$, In most cases, $a = 1$.
- For each repeated linear factor $(ax + b)$, set the fraction above equal to $\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \cdots$, In most cases, $a = 1$.
- For each distinct quadratic factor $(ax^2 + bx + c)$, set the fraction above equal to $\frac{Ax+B}{a_1 x^2 + b_1 x + c_1} + \frac{CX+D}{a_2 x^2 + b_2 x + c_2} + \cdots$, In most cases, $a = 1$ and $b = 0$.
- For each repeated quadratic factor $(ax^2 + bx + c)$, set the fraction above equal to $\frac{Ax+B}{(ax^2+bx+c)} + \frac{CX+D}{(ax^2+bx+c)^2} + \cdots$, In most cases, $a = 1$ and $b = 0$.

Secondly

- Combine the partial fractions over a common denominator and equate the numerator of this expression to the numerator of the original rational function.
- Solve the resulting system of linear equations to find the unknown coefficients.
- Integrate.

Common Antiderivatives

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

$$\int x^{-1}\,dx = \int \frac{1}{x}dx = ln|x| + C$$

$$\int \frac{ax+b}{cx+d}dx = \frac{(bc - a\,d)\,ln|cx+d|}{c^2} + \frac{ax}{c} + C\ , c \neq 0$$

$$\int a^{kx}\,dx = \frac{a^{kx}}{kln(a)} + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin(ax)\,dx = \frac{-\cos(ax)}{a} + C$$

$$\int \cos(ax)\,dx = \frac{\sin(ax)}{a} + C$$

$$\int \frac{1}{1+\mathrm{x}^2}\mathrm{dx} = \mathrm{arctan(x)} + C$$

$$\int \frac{1}{1+ax^2}dx = \frac{\arctan(\sqrt{a}x)}{\sqrt{a}} + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$$

$$\int -\frac{1}{\sqrt{1-x^2}}dx = \arccos(x) + C$$

$$\int \csc^2(x)\,dx = -\cot(x) + C$$

$$\int \sec(x)\,dx = ln|\sec(x) + \tan(x)| + C$$

$$\int \csc(x)\,dx = ln|\csc(x) - \cot(x)| + C$$

$$\int \tan(x)\,dx = ln\,|\sec(x)| + C$$

$$\int \cot(x)\,dx = ln\,|\sin(x)| + C$$

$$\int 0\,dx = C$$

$$\int ln(x)\,dx = x\,ln(x) - x + C$$

$$\int |x|\,dx = \frac{|x|x}{2} + C$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$$

$$\int \frac{1}{ax+b}dx = \frac{ln|ax+b|}{a} + C$$

$$\int \sin^2(x)\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

$$\int \cos^2(x)\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$$

$$\int log_a(x)d\,x = x\,log_a(x) - \frac{x}{ln(a)} + C$$

$$\int \arccos(x)\,dx = x\arccos(x) - \sqrt{1-x^2} + C$$

$$\int \arccos(x)\,dx = x\arcsin(x) + \sqrt{1-x^2} + C$$

Improper Integrals

For an integral $\int_a^b f(x)\,dx$ to be considered improper, there are two possibilities.

1. First, either $a \text{ or } b$ is $\pm\infty$ (Type 1)
   $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$, where $a < c < b$

2. Second, there is a discontinuity in the domain of the function either at $x = a, x = b$ or within (a, b). There can be multiple discontinuities.

More precisely:

- If $f(x)$ is continuous on $[a, b)$ and discontinuous at $x = b$: $\int_a^b f(x)\,dx = \lim_{k\to b^-} \int_a^k f(x)\,dx$, provided the limit exists.

- If $f(x)$ is continuous on $(a, b]$ and discontinuous at $x = a$: $\int_a^b f(x)\,dx = \lim_{k\to a^+} \int_k^b f(x)\,dx$, provided the limit exists.

- If $f(x)$ is discontinuous at $x = c$, where $a < c < b$ and $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent, then: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. Note that both integrals must be convergent. If at least one integral is divergent, all the integral is divergent.

- If $f(x)$ is discontinuous at $x = a$ and $x = b$ and if $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. Again, this requires both integrals to be convergent for this integral to be convergent as well.

Types of possible discontinuities a function can have in the interval $[a, b]$ (This list is not exhaustive).

- A negative square root
- A logarithm smaller or equal to zero
- A division by zero

P-Integrals

If $a > 0$, $\displaystyle\int_a^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$ and is equal to: -$\frac{a^{1-p}}{1-p}$. If $p \leq 1$, the integral diverges

For $a > 0$, If $p < 1$, the integral $\displaystyle\int_0^{a} \frac{1}{x^p}\,dx$ converges and is equal to: $\frac{a^{1-p}}{1-p}$. If $p \geq 1$, the integral diverges.

The Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous on the interval $[a, b]$ and a function $F(x)$ is defined as: $F(x) = \int_a^x f(t)\,dt$

Then: $F'(x) = f(x)$ over $[a, b]$

Given Integral	How to solve it
$F(x) = \int_a^x f(t)\,dt$	$F'(x) = f(x)$
$F(x) = \int_x^a f(t)\,dt$	$F'(x) = -f(x)$
$F(x) = \int_a^{g(x)} f(t)\,dt$	$F'(x) = f\big(g(x)\big) \cdot g'(x)$
$F(x) = \int_{g(x)}^a f(t)\,dt$	$F'(x) = -f\big(g(x)\big) \cdot g'(x)$
$F(x) = \int_{g(x)}^{h(x)} f(t)\,dt$	$\int_{g(x)}^{h(x)} f(t)\,dt = \int_{g(x)}^a f(t)\,dt + \int_a^{h(x)} f(t)\,dt = f\big(h(x)\big) \cdot h'(x) - f\big(g(x)\big) \cdot g'(x)$

The Fundamental Theorem of Calculus, Part 2

$\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$. $F(x)$ is any antiderivative of $f(x)$.

Properties of definite integrals

$$\int_a^b c\,dx = c(b-a)$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x \pm \int_c^b f(x)\,\mathrm{d}x \text{ if } a < c < b$$

$$\int_{-a}^a f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x \text{ if } f(x) \text{ is even}$$

$$\int_{-a}^a f(x)\,\mathrm{d}x = 0 \text{ if } f(x) \text{ is odd}$$

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$$

Special Integrals – Absolute Values

If we have an integral of the form $\int_a^b |f(x)|\,\mathrm{d}x$:

- Solve the integral for when the function inside the absolute value changes sign (when $f(x) = 0$ or has a potential asymptote.
- Plot a number line and identify where on $[a, b]$ the function is $f(x) < 0$.
- Split the integral on the points found above and put a minus sign when the function is smaller than zero.
- Integrate each piece.

Example:

$$\int_1^5 |x^2 - 5x + 6|\,dx$$

$x^2 - 5x + 6 = 0$ when $x = 2$ or $x = 3$

Now, we will always test the number of points we found plus one. In this case, we will test three numbers: One smaller than 2, one between 2 and 3(excluding 2 and 3) and one larger than 3. They can be **any** number. They do not have to be within the bounds of integration.

Say we pick $x = 0, x = 2.5, x = 4$

Now test $f(x)$ for those numbers:

$f(0) = 6$, $f(2.5) = -0.25$, $f(4) = 2$

With that information we complete our number line.

Our integral becomes then: $\int_1^5 |x^2 - 5x + 6|\,dx =$

$$\int_1^2 (x^2 - 5x + 6)\,dx - \int_2^3 (x^2 - 5x + 6)\,dx + \int_3^5 (x^2 - 5x + 6)\,dx = \frac{5}{6} - \left(-\frac{1}{6}\right) + \frac{28}{6} = \frac{34}{6}$$

Volumes, Surface area of Revolution, Arc length and Areas between Curves

Volumes

Axis of revolution is **vertical** (y-axis or x=k, where k is a constant)

	Disks	Washers	Shells
$y - axis$ or $x = 0$	$\int_a^b \pi(f(y))^2\,dy$	$\int_a^b \pi(f(y))^2 - \pi(g(y))^2\,dy$	$\int_a^b 2\pi x[f(x) - g(x)]\,dx$
$x = k$	/	$\int_a^b \pi(k - g(y))^2 - \pi(k - f(y))^2\,dy$	$\int_a^b 2\pi(k - x)[f(x) - g(x)]\,dx$
$x = -k$	/	$\int_a^b \pi(k + f(y))^2 - \pi(k + g(y))^2\,dy$	$\int_a^b 2\pi(k + x)[f(x) - g(x)]\,dx$

Axis of revolution is **horizontal** (x-axis or y=k, where k is a constant)

	Disks	Washers	Shells
$x - axis$ or $y = 0$	$\int_a^b \pi(f(x))^2\,dx$	$\int_a^b \pi(f(x))^2 - \pi(g(x))^2\,dx$	$\int_a^b 2\pi x[f(y) - g(y)]\,dx$
$y = k$	/	$\int_a^b \pi(k - g(x))^2 - \pi(k - f(x))^2\,dx$	$\int_a^b 2\pi(k - y)[f(y) - g(y)]\,dy$
$y = -k$	/	$\int_a^b \pi(k + f(x))^2 - \pi(k + g(x))^2\,dx$	$\int_a^b 2\pi(k + y)[f(y) - g(y)]\,dy$

Surface area of revolution

Use $S = \displaystyle\int_a^b 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ if:

Function is in the form $y = f(x)$ and the function is rotating around the $y-$ axis, in the interval $[a, b]$.

Use $S = \displaystyle\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ if:

Function is in the form $y = f(x)$ and the function is rotating around the $x-$ axis, in the interval $[a, b]$. Replace the y in the integral by $f(x)$, everything should be in terms of x.

Use $S = \displaystyle\int_a^b 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ if:

Function is in the form $x = f(y)$ and the function is rotating around the $y-$ axis, in the interval $[a, b]$. Replace the x in the integral by $f(y)$, everything should be in terms of y.

Use $S = \displaystyle\int_a^b 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ if:

Function is in the form $x = f(y)$ and the function is rotating around the $x-$ axis, in the interval $[a, b]$.

Arc length

If function is in the form $y = f(x)$ on $[a, b]$: $L = \int_a^b \sqrt{1+(f'(x))^2}\,dx = \displaystyle\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

If function is in the form $x = f(y)$ on $[a, b]$: $L = \int_a^b \sqrt{1+(f'(y))^2}\,dx = \displaystyle\int_a^b \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

Area between curves

Vertical Slicing

For two functions of the form $y = f(x)$ and $y = g(x)$, where $f(x) > g(x)$ ($f(x)$ is above on $[a, b]$):

Area = $\int_a^b [f(x) - g(x)]\, dx$

Horizontal Slicing

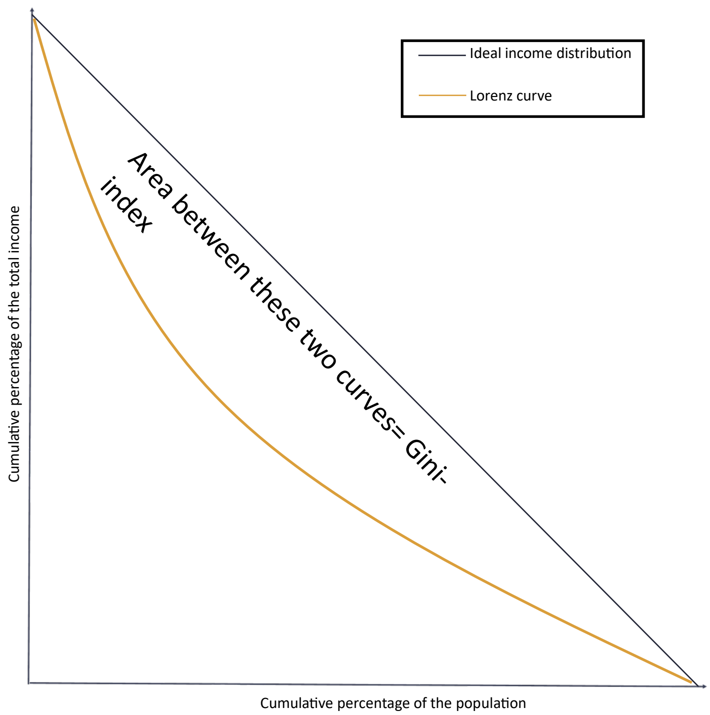
For two functions of the form $x = f(y)$ and $x = g(y)$, where $f(y) > g(y)$ ($f(y)$ is on the right on $[a, b]$:)

Area = $\int_a^b [f(y) - g(y)]\, dy$

Lorenz curve, Consumer and Producer Surplus

Lorenz curve

Used to study income inequality.



Interpretation:

Gini coefficient of zero – Perfectly equal income distribution.

Gini coefficient of one – Perfectly unequal income distribution.

Formula: $G = 2\int_0^1 (x - L(x))\,dx$

As the Gini coefficient increases, the more unequal income distribution is in that nation.

Consumer and Producer Surplus

$S(q)$- Supply curve

$D(q)$- Demand curve

P_m- Market price, first find the right q , denoted as E_q or equilibrium quantity by setting $D(q) = S(q)$. Then plug in the found E_q in $D(q)$ or $S(q)$.

Consumer Surplus: $\int_0^{E_q}(D(q) - P_m)\, dq$

Producer Surplus: $\int_0^{E_q}(P_m - S(q))\, dq$

Total Surplus = Consumer Surplus+ Producer Surplus

Separable Differential equations - Exponential growth and decay, Logistic Model

Separable differential equations framework

Goal: Then integrate both sides (with their respective variables) after having "separated" the x's and the y's and either find a general or a particular solution.

$f(x)\,dx = g(y)\,dy \rightarrow \int f(x)\,dx = \int g(y)\,dy \rightarrow F(x) + C = G(y)$, where $F(x)$ and $G(y)$ respectively are the antiderivatives of $f(x)$ and $g(y)$.

Exponential Growth and Decay

Exponential Growth

$P(t) = P_0 e^{kt}$, where P_0 is the initial condition (usually a population). Can also be given in terms of a separable differential equation.

To find the growth rate k, when a population $n-$ tuples after $m-$ "units of time" (usually years):

$$k = \frac{ln(n)}{m}$$

For example:

Doubles in five years: $k = \frac{ln(2)}{5}$

Quadruples in ten years: $k = \frac{ln(4)}{10}$

Exponential Decay

Exponential Decay is very similar to exponential growth, except that it is usually used to model things like a decaying substance after t years. The k must be negative.

Half-Life case: If the half life of a substance is T years, then:

$$k = \frac{ln(0.5)}{T}$$

Another more general case: If a fraction $\frac{1}{n}$ of the substance remains after T years, then:

$$k = \frac{ln\left(\frac{1}{n}\right)}{T}$$

For example:

25% of a substance remains after five years: $25\% = \frac{1}{4} \rightarrow k = \frac{ln(\frac{1}{4})}{5}$. It is not necessary to transform 25% into a fraction, $\frac{ln(25\%)}{5}$ yields the same result.

Logistic Differential Equation

$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \rightarrow$ Given the initial condition $P(0) = P_0$: $P(t) = \frac{M}{1+\left(\frac{M-P_0}{P_0}\right)e^{-kt}}$

This model is also used to model population growth, however, unlike the exponential growth model, this model imposes a restrain.

k: Growth rate

M: This is the carrying capacity of the environment. It represents the maximum population size that the environment can sustain indefinitely due to limited resources like food, space, and other environmental factors.

Riemann Sums as a Definite Integral

Estimating the area under a curve using n rectangles

$$\Delta x = \frac{b-a}{n}$$

$$x_i^* = a + i\Delta x$$

Left endpoint: $L_n = \sum_{i=0}^{n-1} \Delta x f(x_i^*)$

Right endpoint: $R_n = \sum_{i=1}^{n} \Delta x f(x_i^*)$

Midpoint: $M_n = \sum_{i=1}^{n} \Delta x f(x_{i-0.5}^*)$

Example: Using four rectangles, calculate the area under $f(x) = x^2$ on $[4,5]$, using the left and right endpoints as well as the midpoint.

$$\Delta x = \frac{5-4}{4} = 0.25$$

$$x_i^* = 4 + 0.25i$$

$$x_{i-0.5}^* = 4 + 0.25(i - 0.5) = 3.875 + 0.25i$$

Left endpoint:

$\sum_{i=0}^{3} 0.25f(4 + 1.25i)$ =$0.25[f(4) + f(4.25) + f(4.5) + f(4.75)] = 19.21875$

Right endpoint:

$\sum_{i=1}^{4} 0.25f(4 + 1.25i)$ =$0.25[f(4.25) + f(4.5) + f(4.75) + f(5)] = 21.46875$

Midpoint:

$\sum_{i=1}^{4} 0.25f(3.875 + 0.25i)$ =$0.25[f(4.125) + f(4.375) + f(4.625) + f(4.875)] =$
20.328125

Finding the exact area using Riemann sums

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i^*)$$

Where:

$$\Delta x = \frac{b-a}{n}$$

$$x_i^* = a + i\Delta x$$

Usually, these kinds of limits will always yield to a rational function involving polynomials of the form:

$$\lim_{n\to\infty} \frac{a_m x^m + a_{m-1} x_+^{m-1} \ldots a_1 x + a_0}{b_k x^k + b_{k-1} x^{k-1} + \cdots b_1 x + b_0}$$

The limit above is equal to:

- $\frac{a}{b}$ if $m = k$
- 0 if $k > m$
- ∞ or -∞ if $k < m$

To solve the limit fast, we use:

$$\sum_{i=1}^{n} 1 = \sum_{i=1}^{n} i^0 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\sum_{1=1}^{n} i\right)^2 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$$

Sequence & Series

Sequences

They are on the form of either: $\{a_1, a_2, \dots, a_{n+1,} \dots\}$; $\{a_n\}$; $\{a_n\}_{n=a}^{\infty}$ ($a = 1$ most of the time but a sequence can start at any value)

Where a_n is the general term of a sequence.

A sequence is increasing if $a_n < a_{n+1}$ for every n

A sequence is decreasing if $a_n > a_{n+1}$ for every n

Whether a sequence is decreasing or increasing, that sequence is called a monotonic sequence.

A sequence is convergent if: $\lim_{n\to\infty} a_n$ exists and is finite. If $\lim_{n\to\infty} a_n$ does not exist or is infinite, we say that the sequence diverges. If $\lim_{n\to\infty}|a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

If a number m exists so that $m \le a_n$ for every n, the sequence is bounded below (in other words, none of the numbers in the sequence can be smaller or equal than m), m is a lower bound and is not unique.

If a number M exists so that $M \ge a_n$ for every n, the sequence is bounded above (in other words, none of the numbers in the sequence can be larger or equal than M), M is an upper bound and is not unique.

A sequence that is bounded above and below is called a bounded sequence.

Important: If a sequence is bounded and monotonic, then it must converge.

Special types of sequences – geometric and arithmetic

Geometric sequences are in the form: $\{ar^n\}_{n=0}^{\infty}$, where r is called the **common ratio** and a is some constant and the first term. This sequence always converges if $-1 < r \le 1$.

The n^{th} term of a geometric sequence is defined as: ar^{n-1}

Arithmetic sequences are in the form of $\{a, a + d, a + 2d, a + 3d, \dots, a + (n-1)d, \dots\}$ where d is called the common difference, d is always equal to $a_{n+1} - a_n$ (two consecutive terms) and a is the first term.

The n^{th} term of an arithmetic sequence is defined as $a + (n-1)d$.

Series

Defining series in terms of sequences

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Then the series $s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$. This called a partial sum.

If $\lim_{n\to\infty} s_n$ exists and is finite, the series is convergent. Divergent otherwise. Note that if the series converges, using this method we can know the exact value it converges to. Not the same when using test for convergence, that just tell us whether a series converges.

Basic test for divergence

If $\lim_{n\to\infty} \sum a_n \neq 0$, the series diverges.

IMPORTANT: This test only says that a series is guaranteed to diverge but does not say anything about convergence. If the series terms do happen to go to **zero**, the series may or may not converge. For instance:

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges

$\sum_{n=1}^{\infty} \frac{1}{n}$ Diverges

But $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} = 0$

Absolute convergence and convergence properties

A series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ also converges.

IMPORTANT: A series that is absolutely convergent will also be convergent, but a series that is convergent may or may not be absolutely convergent.

If $\sum a_n$ converges and $\sum |a_n|$ diverges, the series is said to be conditionally convergent.

Given $\sum a_n$ and $\sum b_n$ and that they both converge:

$\sum c a_n = c \sum a_n$

$\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$

$\sum a_n \sum b_n \neq \sum a_n b_n$

Special type of series

Geometric

Just like a geometric sequence, there also exists a geometric series. This is one of the only series that we can find the actual convergence value, of course if it does converge.

They have the following form: $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=0}^{\infty} ar^{n}$ (they are both equal).

The partial sum s_n (the sum of the first n terms) is: $s_n = a\frac{1-r^n}{1-r}$, as mentioned earlier, r is called the **common ratio.**

Provided that $-1 < r < 1$, $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{n} = \frac{\boldsymbol{a}}{\boldsymbol{1-r}}$

Telescoping

A series of the form $\sum_{n=0}^{\infty} \frac{a_m n^m + a_{m-1}n^{m-1} + \dots a_1 n + a_0}{b_k n^k + b_{k-1}n^{k-1} + \dots b_1 n + b_0}$, the denominator and numerator are polynomials with $\boldsymbol{k > m}$. The series converge if we can express the rational expression with terms that cancel each other. For example: $\sum_{n=1}^{\infty} \frac{1}{42+13n+n^2} = \sum_{n=1}^{\infty} (\frac{1}{n+6} - \frac{1}{n+7}) = (\frac{1}{7} - \frac{1}{8}) + (\frac{1}{8} - \frac{1}{9}) + (\frac{1}{9} - \frac{1}{10}) + \cdots$

As we see all numbers simplify except the first term, so this series converges to $\frac{1}{7}$.

What if the series started at $n = 0$ instead?
Then we can use the fact that: $\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n$

In our case, $a_0 = \frac{1}{6} - \frac{1}{7} = \frac{1}{42}$. So $\sum_{n=0}^{\infty} (\frac{1}{n+6} - \frac{1}{n+7})$ would converge to $\frac{1}{42} + \frac{1}{7} = \frac{1}{6}$

Harmonic

For now, only know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the harmonic series and is divergent.

Tests for Convergence

Reminder: These are only test for convergence, they do not tell us what the series converges to, they only tell us that the series converges.

Integral test

If f is **continuous**, **positive** and a **decreasing function** on $[k, \infty)$, and let $a_n = f(n)$:

If $\int_k^\infty f(x)\,dx$ converges, then $\sum_{n=k}^\infty a_n$ converges.

If $\int_k^\infty f(x)\,dx$ diverges, then $\sum_{n=k}^\infty a_n$ diverges.

Example:

Determine the convergence or divergence of $\sum_{n=2}^{\infty} \frac{1}{nln(n)}$

Since $f(n) = \frac{1}{nln(n)}$ is continuous, decreasing and positive on $[2, \infty)$, the test can be applied.

We can solve $\int_2^\infty \frac{1}{xln(x)} dx$ by setting $u = ln(x)$, then $du = \frac{1}{x} dx$. The indefinite integral is equal to $\int\ ln(u)du = ln(ln(x))$. With the bounds: $\lim_{a\to\infty} ln(ln(x))|_2^a = \lim_{a\to\infty} ln(ln(a)) - ln(ln(2)) = \infty$. The integral is divergent then so is the series.

P-series test for convergence

If $k > 0$, then $\sum_{n=k}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. The harmonic series above is a p-series with $p = 1$, since $1 \leq 1$, the series diverges.

Comparison test for convergence

Assume $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\sum b_n$ converges and $a_n \leq b_n$ for all n, then $\sum a_n$ converges.

If $\sum b_n$ diverges and $a_n \geq b_n$ for all n, then $\sum a_n$ diverges.

Example:

Determine the convergence or divergence of $\sum_{n=1}^{\infty}\left(\frac{n}{2n+6}\right)^{n}$

Usually, we use either the geometric series or p-series. The original series **diverges** if $a_n \geq b_n$ with both series being positive and b_n diverges. Note that if $a_n < b_n$, the test is inconclusive.

The original series **converges** if $a_n \leq b_n$ and both series are positive and b_n converges. Note that if $a_n > b_n$, the test is inconclusive.

We can confidently say that $\left(\frac{n}{2n+6}\right)^{n} \leq \left(\frac{n}{2n}\right)^{n}$ for all n and $\left(\frac{n}{2n}\right)^{n} = \left(\frac{1}{2}\right)^{n}$.

So $b_n = \left(\frac{1}{2}\right)^{n}$, is a geometric series with $r = \frac{1}{2}$, since $\left|\frac{1}{2}\right| < 1$, b_n converges.

Now we need to show that $0 \leq a_n \leq b_n$. You can do this by arguing that since the denominator on a_n (the original series) is larger, then a_n is smaller or equal than b_n. A more formal proof can be done using derivatives.

<u>Limit comparison test for convergence</u>

Assume $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$, where $0 < c < \infty$

Then either series converge, or both diverge.

Example: Determine the convergence or divergence of $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}+4\sqrt{n}}$

The original series $\sum a_n$ will **diverge** if $a_n > 0$ and $b_n > 0$, as well as $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum b_n$ diverges.

The original series $\sum a_n$ will **converge** if $a_n > 0$ and $b_n > 0$, as well as $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum b_n$ converges.

Again, usually, we use either the geometric series or p-series as $\sum b_n$.

Let $b_n = \frac{1}{4\sqrt{n}}$, this term carries much more weight than $\sqrt[3]{n}$ when n gets very large. $\sum_{n=1}^{\infty}\frac{1}{4\sqrt{n}} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^{0.5}}$, this is a p-series with $p = 0.5 \leq 1$, therefore, $\sum b_n$ diverges.

Now solving $\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n}+4\sqrt{n}}}{\frac{1}{4\sqrt{n}}}$ (note that $a_n > 0$ and $b_n > 0$).

$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{n}+4\sqrt{n}}}{\frac{1}{4\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}+4\sqrt{n}} \cdot \frac{4\sqrt{n}}{1} = \lim_{n\to\infty} \frac{4\sqrt{n}}{\sqrt[3]{n}+4\sqrt{n}}$. At this step, divide both the denominator and the numerator by $\sqrt{n}$. We are left with: $\lim_{n\to\infty} \frac{4}{n^{-\frac{1}{6}}+4} = \lim_{n\to\infty} \frac{4}{\frac{1}{n^{\frac{1}{6}}}+4} = \frac{4}{0+4} = 1$. Since $L > 0$ and b_n diverges, a_n diverges.

Alternating series test for convergence

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ or equally $\sum_{n=0}^{\infty}(-1)^{n}a_n$, $a_n > 0$

Is a decreasing sequence in a_n ($a_{n+1} \leq a_n$) for all n and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ and equally $\sum_{n=0}^{\infty}(-1)^{n}a_n$ converge.

Example: Determine the convergence or divergence of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{6}{4n+2}$. First, note that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{6}{4n+2} = \sum_{n=0}^{\infty}(-1)^{n}\frac{6}{4n+2}$.

The series will converge if $\sum_{n=1}^{\infty}\frac{6}{4n+2}$ is a decreasing sequence and positive, which is the case. The larger the n, the smaller the terms.

Since $\lim_{n\to\infty}\frac{6}{4n+2} = 0$, the series converge.

Ratio test for convergence

This is a super common test when there are factorials in the series.

Let $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

If $L > 1$, the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If $L = 1$, this test is inconclusive, and we cannot say anything about the convergence of the series.

Example: Determine the convergence or divergence of $\sum_{n=1}^{\infty}\frac{n^n}{n!}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{n^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{\frac{n^n\cdot n}{n!\,(n+1)}}{\frac{n^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{n^n\cdot n}{n!\,(n+1)}\cdot\frac{n!}{n^n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1$$

The test is inconclusive.

<u>Root test for convergence</u>

Let $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$

If $L > 1$, the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

If $L = 1$, this test is inconclusive, and we cannot say anything about the convergence of the series.

Example: Determine the convergence or divergence of $\sum_{n=1}^{\infty}\left(\frac{4n^4+2n^3+8}{\sqrt{7n^6+5n^3-12}}\right)^n$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{4n^4+2n^3+8}{\sqrt{7n^6+5n^3-12}}\right)^n\right|} = \lim_{n\to\infty}\left|\left(\frac{4n^4+2n^3+8}{\sqrt{7n^6+5n^3-12}}\right)^n\right|^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left|\left(\frac{4n^4+2n^3+8}{\sqrt{7n^6+5n^3-12}}\right)\right| = \lim_{n\to\infty}\left|\left(\frac{n^4(4+\frac{2}{n}+\frac{8}{n^4})}{\sqrt{n^6(7+\frac{5}{n^3}-\frac{12}{n^6})}}\right)\right|$$

$$= \lim_{n\to\infty}\frac{n^4(4+\frac{2}{n}+\frac{8}{n^4})}{\sqrt{n^6}\sqrt{7+\frac{5}{n^3}-\frac{12}{n^6}}} = \lim_{n\to\infty}\frac{n^4(4+\frac{2}{n}+\frac{8}{n^4})}{n^3\sqrt{7+\frac{5}{n^3}-\frac{12}{n^6}}} = \lim_{n\to\infty} n\frac{(4+\frac{2}{n}+\frac{8}{n^4})}{\sqrt{7+\frac{5}{n^3}-\frac{12}{n^6}}}$$

$$= \infty\frac{4+0+0}{\sqrt{7+0-0}} = \infty$$

This series diverges.

Power series

A power series is in the form $\sum_{n=0}^{\infty} c_n(x-a)^n$

All functions can be expressed as a power series.

Recall that for a geometric series, the following is true $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for $|x| < 1$.

Example, if we want to express the following function: $\frac{2x}{4+x^2}$ as a representation of a power series: We must manipulate: $\frac{2x}{4+x^2}$ until it has the form of $\frac{1}{1-x}$.

$$\frac{2x}{4+x^2} = 2x\frac{1}{4+x^2} = 2x\frac{1}{4(1+\frac{x^2}{4})} = \frac{x}{2}\frac{1}{(1+\frac{x^2}{4})} = \frac{x}{2}\frac{1}{(1-(-\frac{x^2}{4}))}$$

Then $\frac{2x}{4+x^2} = \sum_{n=0}^{\infty} \frac{x}{2}\left(-\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty} \frac{x^1}{2^1}\left(-\frac{x^2}{2^2}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{x^1}{2^1}\left(\frac{x^2}{2^2}\right)^n =$
$\sum_{n=0}^{\infty}(-1)^n \frac{x^1}{2^1}\left(\frac{x^{2n}}{2^{2n}}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2^{2n+1}}$

After having converted our function to series, we can find the radius of convergence $a < x < b$. Even after finding said radius, it does not mean that the whole series is converging, only in the given radius. The radius of convergence is $R = \frac{b-a}{2}$.

In the example above, we set $a_n = (-1)^n \frac{x^{2n+1}}{2^{2n+1}}$, and $a_{n+1} = (-1)^{n+1} \frac{x^{2n+3}}{2^{2n+3}}$ .Using the ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}.\frac{x^{2n+3}}{2^{2n+3}}}{(-1)^n.\frac{x^{2n+1}}{2^{2n+1}}}\right|$. Skipping the algebra, this simplifies to: $\lim_{n\to\infty}\left|-\frac{x^2}{4}\right|$. Since the n is not present anymore, we can safely remove the limit. Also, since there is an absolute value, we can also remove the minus sign.

So $L = \left|\frac{x^2}{4}\right|$. The ratio test tells us that the series is convergent only if $L < 1$.

$\left|\frac{x^2}{4}\right| < 1 \Rightarrow |x^2| < 4 \Rightarrow -2 < x < 2$. The radius of convergence is then: $R = \frac{2-(-2)}{2} = 2$

Once we have found the radius of convergence, we can now find the interval of convergence. To do so, we also must test the endpoints to see if we include or exclude certain bounds.

First, if $x = -2$: $\sum_{n=0}^{\infty} (\text{-}1)^n \cdot \frac{(\text{-}2)^{2n+1}}{2^{2n+1}} = \sum_{n=0}^{\infty} (\text{-}1)^n (\text{-}1)^{2n+1} \cdot \frac{(2)^{2n+1}}{2^{2n+1}} =$ $\sum_{n=0}^{\infty} (\text{-}1)^{3n+1} = -\sum_{n=0}^{\infty} (\text{-}1)^n$. This is a geometric series with $r = -1$. The geometric series converges if $|r| < 1$, since $|r| < 1 \rightarrow |-1| < 1 \rightarrow 1 < 1$. Since 1 is not less than 1, the series diverges, which means it's divergent at the endpoint if $x = -2$.

Now, let's test $x = 2$: $\sum_{n=0}^{\infty} (\text{-}1)^n \cdot \frac{2^{2n+1}}{2^{2n+1}} = \sum_{n=0}^{\infty} (\text{-}1)^n$. This is the same case for above, which means it's divergent at the endpoint if $x = 2$.

Therefore, our interval of convergence is : $-2 < x < 2$.

We do not need to test the bounds if we only need to find the ratio of convergence.